

Gauge Invariant Variables for Spontaneously Broken $SU(2)$ Gauge Theory in the Spherical Ansatz

Edward Farhi*

*Center for Theoretical Physics
Laboratory for Nuclear Science and
Department of Physics
Massachusetts Institute of Technology
Cambridge, MA 02139*

Krishna Rajagopal†

*Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138*

Robert Singleton, Jr.‡

*Department of Physics
Boston University
Boston, MA 02215*

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†rajagopal@huhepl.harvard.edu. Junior Fellow, Harvard Society of Fellows. Research supported in part by the Milton Fund of Harvard University and by the National Science Foundation under grant PHY-92-18167.

‡bobs@cthulu.bu.edu. Research supported in part by the D.O.E. under contract #DE-FG02-91ER40676 and by an NSF Fellowship under grant number ASC-940031.

Abstract

We describe classical solutions to the Minkowski space equations of motion of $SU(2)$ gauge theory coupled to a Higgs field in the spatial spherical ansatz. We show how to reduce the equations to four equations for four gauge invariant degrees of freedom which correspond to the massive gauge bosons and the Higgs particle. The solutions typically dissipate at very early and late times. To describe the solutions at early and late times, we linearize and decouple the equations of motion, all the while working only with gauge invariant variables. We express the change in Higgs winding of a solution in terms of gauge invariant variables.

I. INTRODUCTION

In this paper, we develop techniques for solving the Minkowski space classical equations of motion for $SU(2)$ gauge theory with spontaneous symmetry breaking introduced via the Higgs mechanism. (This model is the standard electroweak theory without the $U(1)$ gauge field and without the fermions.) We work in the spatial spherical ansatz, the equations for which were first written down by Ratra and Yaffe [1]. Typical solutions to these equations of motion represent inward and outward going spherical shells. The Ratra Yaffe equations are six equations for six functions of r and t . Using the residual $U(1)$ gauge invariance which is present in the spherical ansatz, we reduce this to four equations for four gauge invariant functions of r and t in a manner which guarantees that Gauss's law is automatically satisfied at all time. We linearize these equations and then find new variables in which the linear equations decouple. Typical solutions to the full equations are well approximated by solutions to the linear equations at early and late times. These solutions can be characterized by their change in Higgs winding, which we write in terms of gauge invariant variables. We hope that the gauge invariant variables we introduce will be useful to others studying classical solutions in the spherical ansatz [2].

In previous work done with V. V. Khoze [3], we considered $SU(2)$ gauge theory with no Higgs field. There, we found that in the spherical ansatz solutions have the property that in the far past and far future they can be described (after multiplying by the appropriate power of r) as spherical shells which propagate without distortion. Here, we include the Higgs field. The classical equations of motion are no longer scale invariant, and have the dispersion characteristic of wave equations for massive fields. Thus, the solutions we consider here are qualitatively different than those considered in Ref. [3].

II. THE SPHERICAL ANSATZ

In this section, we review the spherical ansatz [1] for $SU(2)$ gauge theory with a Higgs field. The action for this theory is

$$S = \int d^4x \left\{ -\frac{1}{2} \text{Tr} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \text{Tr} (D^\mu \Phi)^\dagger D_\mu \Phi - \frac{\lambda}{4} (\text{Tr} \Phi^\dagger \Phi - v^2)^2 \right\}, \quad (2.1)$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \\ D_\mu \Phi &= (\partial_\mu - igA_\mu)\Phi \end{aligned} \quad (2.2)$$

with $A_\mu = A_\mu^a \sigma^a/2$ and where the 2×2 matrix Φ is related to the Higgs doublet φ by

$$\Phi(\mathbf{x}, t) = \begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix}. \quad (2.3)$$

Following Ref. [1] we use the metric $ds^2 = -dt^2 + d\mathbf{x}^2$. Our definition of v follows standard conventions, and is $\sqrt{2}$ times the v of Ref. [1].

The spherical ansatz is given by expressing the gauge field A_μ and the Higgs field Φ in terms of six real functions $a_0, a_1, \alpha, \gamma, \mu$ and ν of r and t :

$$\begin{aligned} A_0(\mathbf{x}, t) &= \frac{1}{2g} a_0(r, t) \boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \\ A_i(\mathbf{x}, t) &= \frac{1}{2g} [a_1(r, t) \boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \hat{x}_i + \frac{\alpha(r, t)}{r} (\sigma_i - \boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \hat{x}_i) + \frac{\gamma(r, t)}{r} \epsilon_{ijk} \hat{x}_j \sigma_k] \\ \Phi(\mathbf{x}, t) &= \frac{1}{g} [\mu(r, t) + i\nu(r, t) \boldsymbol{\sigma} \cdot \hat{\mathbf{x}}], \end{aligned} \quad (2.4)$$

where $\hat{\mathbf{x}}$ is the unit three-vector in the radial direction and $\boldsymbol{\sigma}$ are the matrices. For the four dimensional fields to be regular at the origin, we require that $a_0, \alpha, a_1 - \alpha/r, \gamma/r$ and ν vanish as $r \rightarrow 0$. Under a gauge transformation of the form $\exp[i\Omega(r, t) \boldsymbol{\sigma} \cdot \hat{\mathbf{x}}/2]$ with $\Omega(0, t) = 0$, configurations in the spherical ansatz remain in the spherical ansatz and continue to satisfy the appropriate boundary conditions at the origin. Thus, the $SU(2)$ gauge theory reduced to the spherical ansatz has a residual $U(1)$ gauge invariance.

In the spherical ansatz the action (2.1) takes the form [1]

$$\begin{aligned} S = \frac{4\pi}{g^2} \int dt \int_0^\infty dr \left[-\frac{1}{4} r^2 f^{\mu\nu} f_{\mu\nu} - (D^\mu \chi)^* D_\mu \chi - r^2 (D^\mu \phi)^* D_\mu \phi \right. \\ \left. - \frac{1}{2r^2} (|\chi|^2 - 1)^2 - \frac{1}{2} (|\chi|^2 + 1) |\phi|^2 - \text{Re}(i\chi^* \phi^2) \right. \\ \left. - \frac{\lambda}{g^2} r^2 \left(|\phi|^2 - \frac{g^2 v^2}{2} \right)^2 \right], \end{aligned} \quad (2.5)$$

where the indices now run over 0 and 1 and

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \quad (2.6a)$$

$$\chi = \alpha + i(\gamma - 1) \quad (2.6b)$$

$$\phi = \mu + i\nu \quad (2.6c)$$

$$D_\mu \chi = (\partial_\mu - i a_\mu) \chi \quad (2.6d)$$

$$D_\mu \phi = (\partial_\mu - \frac{i}{2} a_\mu) \phi . \quad (2.6e)$$

The notation is chosen to manifest the $U(1)$ gauge invariance present in the action (2.5). The complex scalar fields χ and ϕ have $U(1)$ charges of 1 and 1/2 respectively, a_μ is the $U(1)$ gauge field, $f_{\mu\nu}$ is the field strength, and D_μ is the covariant derivative. The indices are raised and lowered with the 1 + 1 dimensional metric $ds^2 = -dt^2 + dr^2$.

The equations of motion for the reduced theory are

$$-\partial^\mu (r^2 f_{\mu\nu}) = i [D_\nu \chi^* \chi - \chi^* D_\nu \chi] + \frac{i}{2} r^2 [D_\nu \phi^* \phi - \phi^* D_\nu \phi] \quad (2.7a)$$

$$\left[-D^2 + \frac{1}{r^2} (|\chi|^2 - 1) + \frac{1}{2} |\phi|^2 \right] \chi = -\frac{i}{2} \phi^2 \quad (2.7b)$$

$$\left[-D^\mu r^2 D_\mu + \frac{1}{2} (|\chi|^2 + 1) + \frac{2\lambda}{g^2} r^2 \left(|\phi|^2 - \frac{g^2 v^2}{2} \right) \right] \phi = i \chi \phi^* . \quad (2.7c)$$

The same equations are obtained either by varying the action (2.1) and then imposing the spherical ansatz or by varying the action (2.5).

III. GAUGE INVARIANT VARIABLES

Equations (2.7) are six real equations for the six degrees of freedom in χ , ϕ , and a_μ . However they can be reduced to four equations for four gauge invariant variables as we now show. First, we write the complex fields χ and ϕ in polar form¹

¹In Ref. [3] the phase of χ was called φ . In this paper we call it θ .

$$\chi(r, t) = -i\rho(r, t) \exp[i\theta(r, t)] \quad (3.1a)$$

$$\phi(r, t) = \sigma(r, t) \exp[i\eta(r, t)] . \quad (3.1b)$$

The variables ρ and σ are gauge invariant, and the $r = 0$ boundary conditions on α , γ and ν imply

$$\rho(0, t) = 1 \quad (3.2a)$$

$$\theta(0, t) = 0 \quad (3.2b)$$

$$\eta(0, t) = n(t)\pi \text{ when } \sigma(0, t) \neq 0 , \quad (3.2c)$$

where $n(t)$ is an integer. The boundary condition (3.2b) should strictly be that $\theta(0, t)$ is an integer multiple of 2π . However, since ρ never vanishes at the origin, θ is constant in time there and we have taken it to vanish. On the other hand, η can change discontinuously at the origin by an integer multiple of π at times when $\sigma(0, t)$ vanishes.

Since we are in $1 + 1$ dimensions the gauge invariant field strength $f_{\mu\nu}$ must be proportional to $\epsilon_{\mu\nu}$ so we can define the gauge invariant variable ψ via

$$r^2 f_{\mu\nu} = -2\epsilon_{\mu\nu} \psi . \quad (3.3)$$

(Here $\epsilon_{01} = +1$). Since the fields χ and ϕ have $U(1)$ charges of 1 and $1/2$ respectively, the phase variable

$$\xi = \theta - 2\eta \quad (3.4)$$

is also gauge invariant. Note that ξ is periodic with period 2π . Also note that ξ is not defined when either ρ or σ vanishes. We now write the equations of motion in terms of the four gauge invariant variables ρ , σ , ψ and ξ .

Using the definitions (3.1), equation (2.7a) becomes

$$\partial^\mu (r^2 f_{\mu\nu}) + 2\rho^2 (\partial_\nu \theta - a_\nu) + r^2 \sigma^2 (\partial_\nu \eta - \frac{1}{2} a_\nu) = 0 , \quad (3.5a)$$

while (2.7b) becomes the two real equations

$$\partial_\mu \partial^\mu \rho - \rho(\partial^\mu \theta - a^\mu)(\partial_\mu \theta - a_\mu) - \frac{1}{r^2}(\rho^2 - 1)\rho - \frac{1}{2}\sigma^2 \rho = -\frac{1}{2}\sigma^2 \cos \xi \quad (3.5b)$$

and

$$\partial^\mu \left[\rho^2 (\partial_\mu \theta - a_\mu) \right] = \frac{1}{2} \rho \sigma^2 \sin \xi , \quad (3.5c)$$

where we have used the definition (3.4) of ξ . Similarly, (2.7c) becomes the two real equations

$$\partial_\mu r^2 \partial^\mu \sigma - r^2 \sigma (\partial^\mu \eta - \frac{1}{2} a^\mu)(\partial_\mu \eta - \frac{1}{2} a_\mu) - \frac{1}{2}(\rho^2 + 1)\sigma - \frac{2\lambda}{g^2} r^2 \left(\sigma^2 - \frac{g^2 v^2}{2} \right) \sigma = -\rho \sigma \cos \xi \quad (3.5d)$$

and

$$\partial^\mu \left[r^2 \sigma^2 (\partial_\mu \eta - \frac{1}{2} a_\mu) \right] = -\rho \sigma^2 \sin \xi . \quad (3.5e)$$

Note that applying ∂^ν to (3.5a) yields twice (3.5c) plus (3.5e). We can thus view (3.5e) as redundant and drop it.

Now, using (3.3) in (3.5a) we obtain

$$2\epsilon_{\mu\nu} \partial^\mu \psi = 2\rho^2 \partial_\nu \theta + r^2 \sigma^2 \partial_\nu \eta - 2a_\nu (\rho^2 + \frac{1}{4} r^2 \sigma^2) . \quad (3.6)$$

Solving (3.6) for a_ν and using $\xi = \theta - 2\eta$ we find

$$a_\nu = 2\partial_\nu \eta + \frac{\rho^2 \partial_\nu \xi - \epsilon_{\mu\nu} \partial^\mu \psi}{\rho^2 + \frac{1}{4} r^2 \sigma^2} , \quad (3.7)$$

from which we obtain the gauge invariant combinations

$$\partial_\nu \theta - a_\nu = \frac{\epsilon_{\mu\nu} \partial^\mu \psi + \frac{1}{4} r^2 \sigma^2 \partial_\nu \xi}{\rho^2 + \frac{1}{4} r^2 \sigma^2} \quad (3.8a)$$

$$\partial_\nu \eta - \frac{1}{2} a_\nu = \frac{1}{2} \frac{\epsilon_{\mu\nu} \partial^\mu \psi - \rho^2 \partial_\nu \xi}{\rho^2 + \frac{1}{4} r^2 \sigma^2} . \quad (3.8b)$$

From (3.3) we have that $\epsilon^{\mu\nu} \partial_\mu a_\nu = 2\psi/r^2$, which using (3.7) gives

$$\partial^\mu \left\{ \frac{\partial_\mu \psi + \rho^2 \epsilon_{\mu\nu} \partial^\nu \xi}{\rho^2 + \frac{1}{4} r^2 \sigma^2} \right\} - \frac{2}{r^2} \psi = 0 . \quad (3.9a)$$

Substituting (3.8) into (3.5b), (3.5c), and (3.5d), we get

$$\partial_\mu \partial^\mu \rho - \frac{\rho (\frac{1}{4} r^2 \sigma^2 \partial_\mu \xi - \epsilon_{\mu\nu} \partial^\nu \psi)^2}{(\rho^2 + \frac{1}{4} r^2 \sigma^2)^2} - \frac{1}{r^2} (\rho^2 - 1) \rho - \frac{1}{2} \rho \sigma^2 + \frac{1}{2} \sigma^2 \cos \xi = 0 \quad (3.9b)$$

$$\partial^\mu \left\{ \frac{\rho^2 (\frac{1}{4} r^2 \sigma^2 \partial_\mu \xi - \epsilon_{\mu\nu} \partial^\nu \psi)}{\rho^2 + \frac{1}{4} r^2 \sigma^2} \right\} - \frac{1}{2} \rho \sigma^2 \sin \xi = 0 \quad (3.9c)$$

$$\partial_\mu r^2 \partial^\mu \sigma - \frac{\frac{1}{4} r^2 \sigma (\rho^2 \partial_\mu \xi + \epsilon_{\mu\nu} \partial^\nu \psi)^2}{(\rho^2 + \frac{1}{4} r^2 \sigma^2)^2} - \frac{1}{2} (\rho^2 + 1) \sigma - \frac{2\lambda}{g^2} r^2 \left(\sigma^2 - \frac{g^2 v^2}{2} \right) \sigma + \rho \sigma \cos \xi = 0. \quad (3.9d)$$

We have succeeded in casting the equations of motion as four equations (3.9) for four gauge invariant variables. To solve these equations we need the boundary conditions at the origin. The condition $\rho(0, t) = 1$ was already given in (3.2a), and from (3.3) we have $\psi(0, t) = 0$. From (3.2b), (3.2c) and (3.4) we find that when σ does not vanish at the origin, $\xi(0, t) = 0 \bmod 2\pi$. When σ vanishes at the origin, ξ is not defined there. The boundary condition on σ is determined by examining the small- r behavior of (3.9) and demanding that solutions be regular at the origin. When σ is non-zero at the origin, the condition $\partial_r \sigma(0, t) = 0$ must be imposed, and when σ vanishes at the origin the constraint on $\partial_r \sigma(0, t)$ is that it be non-zero and finite. Equations (3.9) can now be solved after specifying initial value data, that is the values of $\rho, \dot{\rho}, \sigma, \dot{\sigma}, \psi, \dot{\psi}, \xi$ and $\dot{\xi}$ at some initial time. With a solution in hand, if a gauge is chosen then the gauge variant variables χ, ϕ and a_μ can be determined using (3.8). (For example, consider $a_0 = 0$ gauge and make a time-independent gauge transformation such that $\eta = 0$ and therefore $\theta = \xi$ at time $t = 0$. Then, with the gauge invariant variables ρ, σ, ψ , and ξ known, the $\nu = 0$ components of equations (3.8) allow one to obtain θ and η at all times. The $\nu = 1$ component of either of equations (3.8) can then be solved for a_1 .) Any initial value data expressed in terms of gauge invariant variables yields, upon choosing a gauge, initial value data in terms of gauge variant variables which is consistent with the Gauss's law constraint. This is because Gauss's law is the $\nu = 0$ component of (3.6).

It is useful to consider the energy functional obtained from the action (2.5). Using Gauss's law, the energy can be written

$$\begin{aligned}
E = \frac{8\pi}{g^2} \int_0^\infty dr \Big[& \frac{1}{2}(\partial_t \rho)^2 + \frac{1}{2}(\partial_r \rho)^2 + r^2 \left(\frac{1}{2}(\partial_t \sigma)^2 + \frac{1}{2}(\partial_r \sigma)^2 \right) + \\
& \frac{\frac{1}{4}r^2 \sigma^2 \rho^2}{\rho^2 + \frac{1}{4}r^2 \sigma^2} \left(\frac{1}{2}(\partial_t \xi)^2 + \frac{1}{2}(\partial_r \xi)^2 \right) + \frac{1}{\rho^2 + \frac{1}{4}r^2 \sigma^2} \left(\frac{1}{2}(\partial_t \psi)^2 + \frac{1}{2}(\partial_r \psi)^2 \right) + \\
& \frac{\psi^2}{r^2} + \frac{1}{4r^2}(\rho^2 - 1)^2 + \frac{1}{4}(\rho^2 + 1)\sigma^2 - \frac{1}{2}\rho\sigma^2 \cos \xi + \frac{1}{2} \frac{\lambda}{g^2} r^2 \left(\sigma^2 - \frac{g^2 v^2}{2} \right)^2 \Big] .
\end{aligned} \tag{3.10}$$

From (3.10) we see that in vacuum, $\rho_{\text{vac}} = 1$, $\sigma_{\text{vac}} = gv/\sqrt{2}$, $\psi_{\text{vac}} = 0$ and $\xi_{\text{vac}} = 0 \bmod 2\pi$. Note that ξ is periodic with period 2π , and there is in fact only one vacuum configuration. The familiar winding number associated with vacua is not seen in gauge invariant variables, because $SU(2)$ vacuum configurations with different winding numbers are gauge transforms of one another, and correspond to a single point in the gauge invariant configuration space described by ρ , σ , ψ , and ξ .

Since ξ is periodic, the reader may be curious whether configurations in which ξ winds by $2\pi n$ are topological solitons. Such finite energy configurations can be constructed, but there is no topological obstruction to their unwinding. These configurations have been studied in some detail by Turok and Zdrozny [4]. Consider a finite energy configuration with $\psi = 0$, $\rho = 1$ and $\sigma = gv/\sqrt{2}$ everywhere, which has $\xi = 0$ for $r < r_1$ and has $\xi = 2\pi$ for $r > r_2$ and in which ξ changes smoothly from 0 to 2π for $r_1 < r < r_2$. For this configuration to unwind, either ρ or σ must vanish, but this costs only a finite energy. Indeed from (3.10), we note that at large radius ρ can vanish at small cost in energy, while at small radius σ can vanish at small cost in energy. Furthermore, because the $\xi = 0$ and $\xi = 2\pi$ vacua are in fact the same configuration, no fields need be changed for r outside $r_1 < r < r_2$ in order to unwind ξ . We conclude that the theory does not have topological solitons of this kind.

IV. THE AMPLITUDE EXPANSION

In this section, we discuss “typical” solutions to the equations of motion in which the amplitudes of the gauge invariant fields (and consequently the energy density) are arbitrarily

small at arbitrarily early and late times. Not all solutions exhibit this behavior. For example, the sphaleron is a static solution and therefore the magnitudes of the fields are constant in time. One can also imagine solutions which are asymptotically equal to the sphaleron for early (late) times but which dissipate into small amplitude configurations at late (early) times. Thus, by restricting ourselves to solutions which dissipate both in the future and the past, we are excluding some solutions from our treatment. For the solutions we wish to treat, at both early and late times it is appropriate to expand the equations of motion as power series in the amplitudes of the fields. At sufficiently early and late times, we need only consider the lowest order (linear) equations whose solutions in fact do dissipate both in the past and the future.

The amplitude expansion is equivalent to a coupling constant expansion, as we now show. First, it is necessary to restore the factors of g which were scaled out in (2.4). This is done by replacing ψ , ρ , σ , and ξ by $g\psi$, $g\rho$, $g\sigma$ and $g\xi$. The vacuum is now given by $\rho = 1/g$, $\sigma = v/\sqrt{2}$, $\psi = 0$ and $\xi = 0$, and it is convenient to define the new parameters

$$m = \frac{1}{2}gv \quad m_H = \sqrt{2\lambda}v \quad \bar{\lambda} = \frac{\lambda}{g^2} , \quad (4.1)$$

and to define the shifted fields y and h by

$$g\rho(r, t) = 1 + g y(r, t) \quad (4.2a)$$

$$g\sigma(r, t) = \sqrt{2}m + \frac{g h(r, t)}{r} . \quad (4.2b)$$

We wish to expand the equations of motion order by order in the fields y , h , ψ and ξ , all of which vanish in vacuum. This is equivalent to performing a power series expansion in g with $\bar{\lambda}$, m , and m_H held fixed, because upon making the substitutions (4.2) and (4.1) in the equations of motion, every y , h , ψ or ξ is multiplied by a single g , and no other g 's occur. In doing an amplitude expansion, it is very helpful to have the equations of motion written in terms of gauge invariant variables. If we were using gauge variant variables, a large amplitude field could carry zero energy. In the formulation we are using, the vacuum is

a single point in the configuration space of the gauge invariant variables, and perturbations described by nonzero y , h , ψ , or ξ must carry energy.

We now expand each of the fields y , h , ψ and ξ in powers of g , expand the equations of motion in g , and keep only the terms linear in g . The linearized equations are

$$\left(\partial^\mu \partial_\mu - m^2 - \frac{2}{r^2}\right) y = 0 \quad (4.3a)$$

$$\left(\partial^\mu \partial_\mu - m_H^2\right) h = 0 \quad (4.3b)$$

$$\partial^\mu \left\{ \frac{\partial_\mu \psi + \epsilon_{\mu\nu} \partial^\nu \xi}{1 + \frac{1}{2} r^2 m^2} \right\} - \frac{2}{r^2} \psi = 0 \quad (4.4a)$$

$$\partial^\mu \left\{ \frac{\frac{1}{2} r^2 m^2 \partial_\mu \xi - \epsilon_{\mu\nu} \partial^\nu \psi}{1 + \frac{1}{2} r^2 m^2} \right\} - m^2 \xi = 0 . \quad (4.4b)$$

We have uncoupled equations for y and h while the ψ and ξ equations remain coupled. The field y satisfies the equation of motion for the angular momentum $l = 1$ partial wave of a field with the W boson mass m , and the field h satisfies the equation of motion for the $l = 0$ partial wave of a field with the Higgs mass m_H .

Now, let us turn to decoupling equations (4.4). Define the new variables $x(r, t)$ and $z(r, t)$ through

$$x = -\frac{2}{r} \psi + \frac{2}{1 + \frac{1}{2} r^2 m^2} \left(\frac{1}{2} r^2 m^2 \dot{\xi} - \psi' \right) \quad (4.5a)$$

$$z = \frac{2}{r} \psi + \frac{1}{1 + \frac{1}{2} r^2 m^2} \left(\frac{1}{2} r^2 m^2 \dot{\xi} - \psi' \right) , \quad (4.5b)$$

where $\dot{} = \partial/\partial t$ and $' = \partial/\partial r$. The linear equations (4.4) for ξ and ψ are equivalent to the decoupled equations

$$\left(-\partial^\mu \partial_\mu + m^2\right) x = 0 \quad (4.6a)$$

$$\left(-\partial^\mu \partial_\mu + m^2 + \frac{6}{r^2}\right) z = 0 . \quad (4.6b)$$

We see that x and z satisfy the equations of motion for $l = 0$ and $l = 2$ partial waves of a field with mass m .

To better understand the angular momentum decomposition it is convenient to return to (2.4), set $g = 1$, write A_i^b in terms of gauge variant variables as

$$A_i^b = (2\alpha + ra_1) \frac{\delta_{ib}}{3r} - \gamma \frac{\epsilon_{ibk} \hat{x}^k}{r} - (\alpha - ra_1) \left(\frac{3\hat{x}_i \hat{x}_b - \delta_{ib}}{3r} \right) , \quad (4.7)$$

and work in $A_0 = 0$ gauge. Note that A_i^b is the sum of terms with $j = 0, 1$, and 2 where j is the sum of isospin, orbital angular momentum and spin. Working to linear order in α , γ , and a_1 , we now relate the terms appearing in (4.7) to the variables x , y , and z . First, from the definitions (2.6b), (3.1a), and (4.2a) we see that to linear order, $\gamma = -y$. Next, from (3.3), and linearizing (3.8a), we obtain

$$\dot{a}_1 = -\frac{2}{r^2} \psi \quad (4.8)$$

$$\dot{\alpha} = \frac{\frac{1}{2}r^2 m^2 \dot{\xi} - \psi'}{1 + \frac{1}{2}r^2 m^2} . \quad (4.9)$$

Comparing with the definitions (4.5), we find

$$2\dot{\alpha} + r\dot{a}_1 = x \quad (4.10a)$$

$$-\gamma = y \quad (4.10b)$$

$$\dot{\alpha} - r\dot{a}_1 = z . \quad (4.10c)$$

Recall that the variables x , y and z satisfy the equations of motion for the partial waves of a field with mass m and angular momentum $l = 0, 1$, and 2 respectively. We now see that they are also associated with $j = 0, 1$, and 2 respectively. Thus, these modes behave as if their angular momentum l is determined by the sum of their orbital angular momentum, spin and isospin [5].

Before solving the linearized equations (4.3) and (4.6), we must specify the $r = 0$ boundary conditions. Using the boundary conditions on ψ , ρ , σ , and ξ presented after (3.9), we see that $x = y = z = h = 0$ at $r = 0$. The initial value data given as ρ , σ , ψ , ξ and their time derivatives at some time are equivalent to initial value data for x , y , z , h , and their time derivatives at that time. (From (4.5), we note that determining the initial values of \dot{x} and \dot{z} requires knowledge of the initial value of $\ddot{\xi}$. This can be obtained from the other initial data using (4.4b).) After solving (4.3) and (4.6) for x , y , z , and h as we describe below, we can use (4.2) and

$$\psi = \frac{r}{6}(2z - x) \quad (4.11a)$$

$$\dot{\xi} = \frac{\psi' + \frac{1}{3}(x + z) \left(1 + \frac{1}{2}r^2 m^2\right)}{\frac{1}{2}r^2 m^2} \quad (4.11b)$$

obtained from (4.5) to determine the solutions (to lowest order in g) for ρ , σ , ψ , and $\dot{\xi}$. Finally, the solution for ξ can be obtained from that for $\dot{\xi}$ using the initial data for ξ .

We must solve the equation

$$\left(\partial_t^2 - \partial_r^2 + m^2 + \frac{l(l+1)}{r^2}\right)F_l(r, t) = 0 \quad (4.12)$$

subject to the boundary condition $F_l(0, t) = 0$. We will then use $l = 0, 1$, and 2 solutions for x , y , and z , and an $l = 0$ solution with m replaced by m_H for h . The general solution to (4.12) is

$$F_l(r, t) = \int_0^\infty \frac{dk}{2\pi} \left\{ f_l(k) kr j_l(kr) \exp(-i\omega_k t) + \text{c.c.} \right\} \quad (4.13)$$

where $\omega_k = \sqrt{k^2 + m^2}$ and $j_l(s)$ satisfies

$$\left(\frac{d^2}{ds^2} + 1 - \frac{l(l+1)}{s^2}\right)s j_l(s) = 0 \quad (4.14)$$

with $\lim_{s \rightarrow 0} s j_l(s) = 0$. The solutions we require are:

$$s j_0(s) = \sin s \quad (4.15a)$$

$$s j_1(s) = -\cos s + \frac{\sin s}{s} \quad (4.15b)$$

$$s j_2(s) = -\sin s - \frac{3 \cos s}{s} + \frac{3 \sin s}{s^2} . \quad (4.15c)$$

Finally, the complex function $f_l(k)$ can be determined from the initial conditions on $F_l(r, t)$ and $\dot{F}_l(r, t)$. So, given initial conditions on x , y , z , and h and their time derivatives, we can now obtain explicit solutions to the linearized equations of motion.

The solutions to the linearized equations look like shells of energy which come in from large r , bounce, and then recede to large r again. Because of the mass term in (4.12) the shells do not maintain their shape as they propagate at large r . As $t \rightarrow \pm\infty$ the solutions disperse and the amplitudes of the fields decrease. This means that for the solutions we are discussing, using the linearized equations is a better and better approximation at larger and larger $|t|$. In the far past, solutions to the full nonlinear equations of motion are well approximated by solutions to the linear equations specified by the functions $f_l^p(k)$, and in the far future they are once again well approximated by solutions to the linear equations specified by different functions $f_l^f(k)$. Working order by order in the amplitude expansion, one could obtain f_l^f perturbatively given f_l^p .

V. Q , ΔN_H , AND THE SPHALERON

In the spherical ansatz, the topological charge

$$Q = \frac{g^2}{32\pi^2} \int_{-\infty}^{\infty} dt \int d^3x \epsilon^{\mu\nu\alpha\beta} \text{Tr}(F_{\mu\nu} F_{\alpha\beta}) . \quad (5.1)$$

can be written [1]

$$Q = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \epsilon^{\mu\nu} \partial_\mu \left[a_\nu + \frac{i}{2} (\chi D_\nu \chi^* - \chi^* D_\nu \chi) \right] . \quad (5.2)$$

The integrand is the divergence of a gauge variant current, and so cannot be written in terms of gauge invariant variables. For solutions to the equations of motion, however, we can use equations (3.1a), (3.3), (3.8a), and (3.9a) to write a gauge invariant current

$$k^\mu(r, t) = -\frac{1}{2\pi} \left[\frac{(1 - \rho^2)(\partial^\mu \psi - \frac{1}{4} r^2 \sigma^2 \epsilon^{\mu\nu} \partial_\nu \xi)}{(\rho^2 + \frac{1}{4} r^2 \sigma^2)} \right] - \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \xi , \quad (5.3)$$

which is zero in vacuum regions of space-time and which satisfies

$$\frac{g^2}{8\pi} r^2 \epsilon^{\mu\nu\alpha\beta} \text{Tr}(F_{\mu\nu} F_{\alpha\beta}) = \partial_\mu k^\mu . \quad (5.4)$$

It is shown in Ref. [6], without reference to the spherical ansatz, that the topological charge of a solution to the equations of motion is not well-defined as an integral over all space-time. Attempting to evaluate Q as the limit of a sequence of integrals taken over larger and larger regions of space-time such that in the limit all of space-time is included, one can obtain different results for different sequences of regions. This observation applies to all solutions which linearize in the far future or in the far past. Thus although one could obtain a finite result upon evaluating (5.2) for a solution, we do not believe that the result would have any significance.

From Ref. [6], we learn that it is profitable to characterize solutions by their change in Higgs winding number. The matrix Φ of (2.3) can be written in terms of a unitary matrix U according to

$$\Phi(\mathbf{x}, t) = (\varphi_1^* \varphi_1 + \varphi_2^* \varphi_2)^{1/2} U(\mathbf{x}, t) \quad (5.5)$$

at all space-time points where $|\varphi| \equiv (\varphi_1^* \varphi_1 + \varphi_2^* \varphi_2)^{1/2}$ is non-vanishing. We will consider only those solutions for which the fields approach their vacuum values in the $|\mathbf{x}| \rightarrow \infty$ limit for all t . Without loss of generality, we can work in a gauge in which the boundary condition

$$\lim_{|\mathbf{x}| \rightarrow \infty} U(\mathbf{x}, t) = 1 \quad (5.6)$$

is satisfied. At any time t when $|\varphi| \neq 0$ throughout space, the configuration can be characterized by the integer valued winding number

$$N_H(t) = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{Tr} \left(U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U \right). \quad (5.7)$$

The Higgs winding number N_H is gauge invariant under small gauge transformations, but large gauge transformations can change it by an integer. For the solutions of interest to us, at early and late times the fields are close to their vacuum values. Therefore, $N_H(t)$ becomes constant in time in the far past and in the far future and we can define the gauge invariant quantity

$$\Delta N_H = \lim_{t \rightarrow \infty} N_H(t) - \lim_{t \rightarrow -\infty} N_H(t). \quad (5.8)$$

We now state the results concerning ΔN_H proved in Ref. [6]. These results were established without reference to the spherical ansatz. First, solutions with $\Delta N_H \neq 0$ which dissipate at early and late times must have at least the sphaleron energy. Second, suppose we couple a quantized chiral fermion to the classical gauge and Higgs backgrounds considered in this paper, giving the fermion a gauge invariant mass via a Yukawa coupling. The number of fermions produced in a background given by a solution which dissipates at early and late times is equal to ΔN_H [7].

In the spherical ansatz, Φ is given by

$$\Phi(\mathbf{x}, t) = \frac{1}{g} \sigma(r, t) \exp [i\eta(r, t) \boldsymbol{\sigma} \cdot \hat{\mathbf{x}}] \quad (5.9)$$

where the boundary condition (5.6) on U can be implemented by working in a gauge in which

$$\lim_{r \rightarrow \infty} \eta(r, t) = 0 . \quad (5.10)$$

The boundary condition (3.2c), namely that $\eta(0, t)$ is an integer multiple of π , ensures that (5.9) makes sense at the origin. At times when $\sigma \neq 0$ for all r , the Higgs winding number (5.7) is given by

$$N_H(t) = \frac{1}{\pi} \int_0^\infty dr \frac{\partial \eta}{\partial r} (1 - \cos 2\eta) = -\frac{\eta(0, t)}{\pi} , \quad (5.11)$$

where we have used both (5.10) and (3.2c). At times when σ is nonzero at the origin, $\eta(0, t)$ is constant and therefore so is $N_H(t)$. The Higgs winding number can change only at times when σ vanishes at $r = 0$.

ΔN_H , unlike $N_H(t)$, is gauge invariant and we now show how to write it in terms of gauge invariant variables in the spherical ansatz. We assume that zeros of σ occur only at isolated points in the (r, t) plane. For $\sigma = (\mu^2 + \nu^2)^{1/2}$ to vanish, both μ and ν must vanish at the same (r, t) point. Typically, the zeroes of μ and ν each define one dimensional curves that intersect at isolated points. So we believe that the case we are treating, for which the zeros of σ are isolated, is generic. From (5.8), we can express the change in Higgs winding as

$$\Delta N_H = -\frac{1}{\pi} \int_C dx^\mu \partial_\mu \eta(r, t) , \quad (5.12)$$

where the contour C is oriented from the infinite past to the infinite future, following $r = 0$ except for infinitesimal excursions to non-zero r to avoid any zeroes of σ which lie along $r = 0$. In this way we ensure that in the generic case all other zeros of σ lie to the right of C , since in this case the zeros of σ are isolated. Furthermore, all zeros of ρ lie to the right of C , since ρ is continuous and equal to unity along $r = 0$. Since neither σ nor ρ vanish on C , both η and θ are defined on C , and therefore $\xi = \theta - 2\eta$ is also defined on C . From the boundary condition (3.2b) on θ , we see that equation (5.12) implies

$$\Delta N_H = \frac{1}{2\pi} \int_C dx^\mu \partial_\mu \xi(r, t) . \quad (5.13)$$

Note that (5.13) is manifestly gauge invariant because ξ is gauge invariant. Furthermore, expression (5.13) is valid for any finite energy sequence of configurations parameterized by t whose zeros of σ are isolated and whose zeros of σ at $r = 0$ are restricted to a finite range of t , and not just for sequences which are solutions to the equations of motion.

Solutions with $\Delta N_H \neq 0$ must have at least one time when $\sigma(0, t) = 0$. (Even outside the spherical ansatz, the Higgs field must vanish at some point in space-time in order for the Higgs winding to change.) We now show that solutions in the spherical ansatz with $\Delta N_H \neq 0$ must have $\rho = 0$ at some point in the (r, t) plane. Because we are discussing solutions with $\xi \rightarrow 0$ for $r \rightarrow \infty$, in a gauge in which (5.10) is satisfied we also have

$$\lim_{r \rightarrow \infty} \theta(r, t) = 0 . \quad (5.14)$$

For the solutions we are considering, the amplitudes of all the fields dissipate at early and late times, and in particular $\xi = \theta - 2\eta$ dissipates. This means that

$$\lim_{t \rightarrow \pm\infty} \int_0^\infty dr \frac{\partial \xi}{\partial r} = 0 , \quad (5.15)$$

which in turn implies that

$$\lim_{t \rightarrow \infty} \frac{1}{2\pi} \int_0^\infty dr \frac{\partial \theta}{\partial r} - \lim_{t \rightarrow -\infty} \frac{1}{2\pi} \int_0^\infty dr \frac{\partial \theta}{\partial r} = \Delta N_H . \quad (5.16)$$

Together with the conditions (3.2b) and (5.14), equation (5.16) implies that θ changes by $2\pi\Delta N_H$ as one traverses a large rectangular path in the (r, t) plane with sides at $r = 0$ and at large r , and constant t sides in the far future and far past. Hence, for a solution to have nonzero ΔN_H , there must be at least one point in the (r, t) plane where ρ vanishes.

The sphaleron configuration of Manton and Klinkhamer [8] is given in terms of our gauge invariant variables by

$$\rho^{\text{sph}} = |2f(r) - 1| \quad (5.17a)$$

$$\sigma^{\text{sph}} = \frac{gv}{\sqrt{2}} h(r) \quad (5.17b)$$

$$\psi^{\text{sph}} = 0 \quad (5.17c)$$

$$\xi^{\text{sph}} = \begin{cases} \pi ; & 0 < r < r_0 \\ 0 ; & r > r_0 \end{cases} \quad (5.17d)$$

where $f(r)$ and $h(r)$ are continuous functions that vary from zero to one as r increases from zero to infinity. As $f(r)$ varies between zero and one, there is a nonzero radius r_0 for which $f(r_0) = 1/2$. On this shell, ρ vanishes and θ is not defined; whereas η is not defined at $r = 0$, where σ vanishes. The variable ξ is not defined at both $r = 0$ and $r = r_0$. In general, a solution with $\Delta N_H \neq 0$ can have its zeros of ρ and σ at different times. In the special case where such a solution goes exactly through the sphaleron configuration at some time t_0 , then it has both a zero of ρ and of σ at that time. In such a case, in the (r, t) plane ξ winds by 2π in one direction about $(0, t_0)$ and in the opposite direction about (r_0, t_0) .

VI. CONCLUSIONS

We have cast the full nonlinear equations of motion in the spherical ansatz as four equations for four gauge invariant degrees of freedom in such a way that the Gauss's law constraint can easily be implemented. We hope these equations are of use for numerical study or for those seeking exact solutions. We have also studied the linear form of these equations, which are the leading term in an amplitude expansion. The linear equations we obtain describe an angular momentum $l = 0$ partial wave of a field with the Higgs mass, and angular momentum $l = 0, 1$, and 2 partial waves of a field with the W boson

mass. These modes behave as if their angular momentum l is determined by the sum of their orbital angular momentum, spin and isospin. Typical solutions to the full equations of motion are well approximated by solutions to the linearized equations at early and late times. Furthermore, the fields approach their vacuum values and the Higgs winding number is constant in the far past and in the far future. The change in Higgs winding, which is related to anomalous fermion production [6,7], can be expressed as an integral involving the gauge invariant variables.

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